Modified Dispersion Relations: from Black-Hole Entropy to the Cosmological Constant.

Remo Garattini*
Università degli Studi di Bergamo, Facoltà di Ingegneria,
Viale Marconi 5, 24044 Dalmine (Bergamo) Italy. and
I.N.F.N. - sezione di Milano, Milan, Italy.

Quantum Field Theory is plagued by divergences in the attempt to calculate physical quantities. Standard techniques of regularization and renormalization are used to keep under control such a problem. In this paper we would like to use a different scheme based on Modified Dispersion Relations (MDR) to remove infinities appearing in one loop approximation in contrast to what happens in conventional approaches. In particular, we apply the MDR regularization to the computation of the entropy of a Schwarzschild black hole from one side and the Zero Point Energy (ZPE) of the graviton from the other side. The graviton ZPE is connected to the cosmological constant by means of of the Wheeler-DeWitt equation.

PACS numbers: 04.60.-m, 04.62+v, 05.10.Cc
Keywords: Black Hole Entropy; Quantum Gravity; Cosmological Constant.

I. INTRODUCTION

The appearance of a trans-Planckian physics in Black Hole thermodynamics has led many authors to consider that some deep change in particle physics should come into play. One realization of these ideas is represented by the modification of the Heisenberg uncertainty relations, better known as *Generalized Uncertainty Principle* (GUP)[1–4]. This principle is based on the following inequality

$$\Delta x \Delta p \geq \hbar + \frac{\lambda_p^2}{\hbar} \left(\Delta p\right)^2, \tag{1}$$

where $\hbar$ is the Planck constant and λ_p is the Planck length. Of course, the above inequality affects the Liouville measure which becomes

$$\frac{d^3x d^3p}{\left(2\pi\hbar\right)^3 \left(1+\lambda p^2\right)^3}. \tag{2}$$

When $\lambda = 0$, the formula reduces to the ordinary counting of quantum states. If Eq.(2) is used for computing the entropy of a black hole from a Quantum Field Theory point of view, the usual UV divergence at the horizon can be removed[1, 2]. Indeed, without introducing Eq.(14) one is forced to use traditional methods for removing divergences: for example renormalizing the Newton constant[5], or using Pauli-Villars regularization[6]. It is clear that the distortion of the Liouville measure plays a key rôle in regularizing divergent integrals. Non-commutative geometry provides another powerful method to have such a distortion. As shown in [7–9], one finds[10]

$$dn = \frac{d^3x d^3k}{\left(2\pi\right)^3} \implies dn = \frac{d^3x d^3k}{\left(2\pi\right)^3} \exp\left(-\frac{\theta}{4}k^2\right). \tag{3}$$

This deformation corresponds to an effective cut off on the background geometry. The UV cut off is triggered only by higher momenta modes $\gtrsim 1/\sqrt{\theta}$ which propagate over the background geometry. When $\theta = 0$, the formula reduces to the ordinary counting of quantum states. An application of non-commutative geometry to the computation of black hole entropy shows that the usual horizon divergence disappears[11]. In connection with these ideas, in recent years, there has been a proposal on how the fundamental aspects of special relativity can be modified at very high energies. This modification has been termed *Doubly Special Relativity* (DSR)[12]. In DSR, the Planck mass is regarded as an observer independent energy scale. One of its effects is that the usual dispersion relation of a massive particle of mass m is modified into the following expression

$$E^2 g_1^2\left(E/E_P\right) - p^2 g_2^2\left(E/E_P\right) = m^2, \tag{4}$$

*Electronic address: remo.garattini@unibg.it.

where $g_1(E/E_P)$ and $g_2(E/E_P)$ are two unknown functions which have the following property

$$\lim_{E/E_P\to 0} g_1(E/E_P) = 1 \qquad \text{and} \qquad \lim_{E/E_P\to 0} g_2(E/E_P) = 1. \tag{5}$$

Thus, the usual dispersion relation is recovered at low energies. Eqs.(4, 5) are a representation of "*Modified Dispersion Relations"* (MDRs). The common motivation in using them is in that they can be used as a phenomenological approach to investigate physics at the Planck scale, where General Relativity is no longer reliable. Moreover, we expect the functions $g_1(E/E_P)$ and $g_2(E/E_P)$ modify the UV behavior of quantum fields in the same way as GUP and Non-commutative geometry do, respectively. Note that GUP and MDR modifications are strictly connected[13]. Since the form of $g_1(E/E_P)$ and $g_2(E/E_P)$ is unknown and they have to obey the property (5), we have a large amount of arbitrariness in fixing the dependence on E/E_P, even if some specific choices have been proposed by G. Amelino-Camelia et al.[14] in the context of black hole thermodynamics. MDRs play a relevant rôle also when the background is curved. Following the analysis of Magueijo and Smolin[15] one can define the following "*rainbow metric*"

$$ds^2 = -\frac{N^2(r)\,dt^2}{g_1^2(E/E_P)} + \frac{dr^2}{\left(1-\frac{b(r)}{r}\right) g_2^2(E/E_P)} + \frac{r^2}{g_2^2(E/E_P)}\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{6}$$

which is a solution of the distorted Einstein's Field equations

$$G_{\mu\nu}(E) = 8\pi G(E)\, T_{\mu\nu}(E) + g_{\mu\nu}\Lambda(E). \tag{7}$$

$G(E)$ is an energy dependent Newton's constant, defined so that $G(0)$ is the physical Newton's constant. Similarly we have an energy dependent cosmological constant $\Lambda(E)$. The function $b(r)$ will be referred to as the "shape function". The shape function may be thought of as specifying the shape of the spatial slices. If the equation $b(r_w) = r_w$ is satisfied for some values of r, then we say that the points r_w are horizons for the metric (6). In this paper we will discuss the use of MDRs on black hole entropy calculation[16] and on the estimation of the cosmological constant computed with the help of a revisited Wheeler-DeWitt equation (WDW)[17]. Units in which $\hbar = c = k = 1$ are used throughout the paper.

II. BLACK HOLE ENTROPY WITH MDRS

To start with, we fix the ideas on a real massless scalar field described by the action

$$I = -\frac{1}{2}\int d^4x\sqrt{-g}\left[g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi\right] \tag{8}$$

in the background geometry of Eq.(6) with $N(r)$ described by

$$N^2(r) = \exp\left(-2A(r)\right)\left(1-\frac{b(r)}{r}\right), \tag{9}$$

where $A(r)$ is known as the "redshift function" that describes how far the total gravitational redshift deviates from that implied by the shape function. Without loss of generality we can fix the value of $A(r)$ at infinity such that $A(\infty) = 0$. The Euler-Lagrange equations are

$$\frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}g^{\mu\nu}\partial_\nu\right)\phi = 0. \tag{10}$$

In order to use the WKB approximation, if ϕ has a separable form, we can define an r-dependent radial wave number $k(r,l,E)$

$$k_r^2(r,l,E) \equiv \frac{1}{\left(1-\frac{b(r)}{r}\right)}\left[\exp\left(2A(r)\right)\frac{E^2h^2(E/E_P)}{\left(1-\frac{b(r)}{r}\right)} - \frac{l(l+1)}{r^2}\right], \tag{11}$$

with

$$h(E/E_P) = \frac{g_1(E/E_P)}{g_2(E/E_P)}. \tag{12}$$

The number of modes with frequency less than E is given approximately by

$$\tilde{g}(E)=\frac{1}{\pi}\int_0^{l_{max}}(2l+1)\int_{r_w}^{R}\sqrt{k^2(r,l,E)}drdl, \tag{13}$$

where it is understood that the integration with respect to r and l is taken over those values which satisfy $r_w \leq r \leq R$ and $k^2(r,l,E) \geq 0$. Thus, from Eq.(11) we get

$$\frac{d\tilde{g}(E)}{dE}=\int\frac{\partial\nu(l,E)}{\partial E}(2l+1)dl=\frac{2}{\pi}\frac{d}{dE}\left(\frac{1}{3}E^3h^3\left(E\right)\right)\int_{r_w}^{R}dr\frac{\exp\left(3A\left(r\right)\right)}{\left(1-\frac{b(r)}{r}\right)^2}r^2. \tag{14}$$

Defining β as the inverse temperature measured at infinity, the free energy is given by

$$F=\frac{1}{\beta}\int_0^{\infty}\ln\left(1-e^{-\beta E}\right)\frac{d\tilde{g}(E)}{dE}dE=F_{r_w}+F_R. \tag{15}$$

We divide the integration range into two intervals: in $[r_w,r_1]$ we define $F_{r_w}$and in $[r_1,+\infty)$ with $r_1>r_w$ we define F_R. Assuming that $A\left(r\right)<\infty$, $\forall r\in[r_w,+\infty)$, F_R is dominated by large volume effects for large R. This term will give the contribution to the entropy of a homogeneous quantum gas in flat space at a uniform temperature T. We fix our attention on

$$F_{r_w}=\frac{2}{\pi}\frac{1}{\beta}\int_0^{\infty}\ln\left(1-e^{-\beta E}\right)\frac{d}{dE}\left(\frac{1}{3}E^3h^3\left(E\right)\right)dE\int_{r_w}^{r_1}drr^2\frac{\exp\left(3A\left(r\right)\right)}{\left(1-\frac{b(r)}{r}\right)^2}. \tag{16}$$

In proximity of r_w

$$1-\frac{b\left(r\right)}{r}=\frac{r-r_w}{r_w}\ \left[1-b'\left(r_w\right)\right] \tag{17}$$

and the radial part of F_{r_w} becomes divergent. This ultraviolet divergence has been cured by 't Hooft, who introduced a "*brick wall* r_0" proportional to l_P^2[18]. Nevertheless, since spacetime is modified by a "*rainbow metric*", it is quite natural that even the "*brick wall*" is affected by this distortion. To see such an effect, we perform the radial integration in F_{r_w} to obtain

$$\int_{r_w+r_0}^{r_1}drr^2\frac{\exp\left(3A\left(r\right)\right)}{\left(1-\frac{b(r)}{r}\right)^2}=\int_{r_w+r(E/E_P)}^{r_1}drr^2\frac{\exp\left(3A\left(r\right)\right)}{\left(1-\frac{b(r)}{r}\right)^2}\simeq r_w^4\frac{\exp\left(3A\left(r_w\right)\right)}{\left(1-b'\left(r_w\right)\right)^2}\frac{1}{r\left(E/E_P\right)}, \tag{18}$$

where we have assumed that, in proximity of the throat the brick wall is no longer a constant but it becomes a function of E/E_P. Plugging Eq.(18) into Eq.(16) we find

$$F_{r_w}=-\frac{2r_w^4}{3\beta\pi}\frac{\exp\left(3\Lambda\left(r_w\right)\right)}{\left(1-b'\left(r_w\right)\right)^2}\int_0^{\infty}E^3h^3\left(E/E_P\right)\frac{d}{dE}\left[\frac{\ln\left(1-\exp\left(-\beta E\right)\right)}{r\left(E/E_P\right)}\right]dE, \tag{19}$$

where we have integrated by parts with the condition that $h\left(E/E_P\right)$ be chosen in such a way to allow the convergence when $E/E_P\to\infty$. Without loss of generality we write

$$r\left(E/E_P\right)=r_w\sigma\left(E/E_P\right), \tag{20}$$

with

$$\sigma\left(E/E_P\right)\to 0,\qquad E/E_P\to 0. \tag{21}$$

In this way the horizon divergence is still present but it is translated in terms of a function of E/E_P. Plugging Eq.(20) into Eq.(19), we obtain

$$F_{r_w}=-\frac{C_{r_w}}{3\beta r_w}\int_0^{\infty}E^3h^3\left(E/E_P\right)\frac{d}{dE}\left[\frac{\ln\left(1-\exp\left(-\beta E\right)\right)}{\sigma\left(E/E_P\right)}\right]dE$$

where we have defined

$$C_{r_w} = \frac{2r_w^4}{\pi} \frac{\exp\left(3\Lambda\left(r_w\right)\right)}{\left(1 - b'\left(r_w\right)\right)^2}. \tag{22}$$

It is clear that $g_1\left(E/E_P\right)$ and $g_2\left(E/E_P\right)$ must be chosen in such a way to compensate the vanishing of $\sigma\left(E/E_P\right)$, otherwise the horizon divergence (*brick wall*) cannot be eliminated. For example, one good candidate for the convergence is

$$h\left(E/E_P\right) = \exp\left(-\frac{E}{E_P}\right). \tag{23}$$

A good candidate, but not exhaustive for $\sigma\left(E/E_P\right)$ is

$$\sigma\left(E/E_P\right) = \left(\frac{E}{E_P}\right)^{\alpha}, \qquad \alpha > 0. \tag{24}$$

In the limit where $\beta E_P \gg 1$, the total energy U is

$$U = \frac{\partial\left(\beta F_{r_w}\right)}{\partial \beta} = r_w^2 \frac{\exp\left(2A\left(r_w\right)\right)}{1 - b'\left(r_w\right)} \frac{2E_P^2}{9\beta^2 \kappa_w} \pi = r_w^2 \frac{\exp\left(2A\left(r_w\right)\right)}{1 - b'\left(r_w\right)} \frac{E_P^2}{9\beta} \tag{25}$$

and the entropy S is

$$S = \beta^2 \frac{\partial F_{r_w}}{\partial \beta} = r_w^2 \frac{\exp\left(2\Lambda\left(r_w\right)\right)}{1 - b'\left(r_w\right)} \frac{4E_P^2}{9\beta\kappa_w} \pi = \frac{A_{r_w} E_P^2}{4} \frac{\exp\left(2\Lambda\left(r_w\right)\right)}{1 - b'\left(r_w\right)} \frac{2}{9\pi}, \tag{26}$$

where we have used the expression for the surface gravity in the low energy limit

$$\kappa_w = \frac{1}{2r_w} \exp\left(-A\left(r_w\right)\right)\left[1 - b'\left(r_w\right)\right] \tag{27}$$

and where we have integrated over E. To recover the area law, we have to impose that

$$\frac{\exp\left(2\Lambda\left(r_w\right)\right)}{1 - b'\left(r_w\right)} = \frac{9\pi}{2} \tag{28}$$

and

$$\frac{1}{\beta} = T = \frac{\kappa_w}{2\pi}. \tag{29}$$

This corresponds to a changing of the time variable with respect to the Schwarzschild time. The total energy becomes

$$U = r_w^2 \frac{\pi E_P^2}{2\beta}, \tag{30}$$

which in terms of the Schwarzschild radius $r_w = 2MG$ and inverse Hawking temperature $\beta = 8\pi MG$ becomes

$$U = 4M^2 G^2 \frac{E_P^2}{16MG} = \frac{M}{4}. \tag{31}$$

Note the discrepancy of a factor of 3/2 with the 't Hooft result.

III. GRAVITY'S RAINBOW AND THE WDW EQUATION

The WDW equation was originally introduced by Bryce DeWitt as an attempt to quantize General Relativity in a Hamiltonian formulation. It is described by[19]

$$\mathcal{H}\Psi = \left[(2\kappa)\, G_{ijkl}\pi^{ij}\pi^{kl} - \frac{\sqrt{g}}{2\kappa}\left({}^3R - 2\Lambda\right)\right]\Psi = 0 \tag{32}$$

and it represents the quantum version of the classical constraint which guarantees the invariance under time reparametrization. G_{ijkl} is the super-metric, π^{ij} is the super-momentum,^3R is the scalar curvature in three dimensions and Λ is the cosmological constant, while $\kappa = 8\pi G$ with G the Newton's constant. In this way, the WDW equation is written in its most general form. The main reason to use such an equation to discuss renormalization problems is related to the possibility of formally re-writing the WDW equation as an expectation value computation. Rather than reproduce the formalism, we shall refer the reader to Refs.[20] for details, when necessary. However, for self-completeness and self-consistency, we present here a brief outline of the formalism used[1]. Multiplying Eq.(32) by $\Psi^*[g_{ij}]$ and functionally integrating over the three spatial metric g_{ij} we find

$$\frac{1}{V}\frac{\int \mathcal{D}[g_{ij}]\,\Psi^*[g_{ij}]\int_\Sigma d^3x\hat{\Lambda}_\Sigma\Psi[g_{ij}]}{\int \mathcal{D}[g_{ij}]\,\Psi^*[g_{ij}]\,\Psi[g_{ij}]} = \frac{1}{V}\frac{\left\langle \Psi \left| \int_\Sigma d^3x\hat{\Lambda}_\Sigma \right| \Psi \right\rangle}{\langle \Psi | \Psi \rangle} = -\frac{\Lambda}{\kappa}. \tag{33}$$

In Eq.(33) we have also integrated over the hypersurface Σ and we have defined

$$V = \int_\Sigma d^3x\sqrt{g}. \tag{34}$$

V is the volume of the hypersurface Σ and

$$\hat{\Lambda}_\Sigma = (2\kappa)\,G_{ijkl}\pi^{ij}\pi^{kl} - \sqrt{g}^3R/(2\kappa)\,. \tag{35}$$

In this form, Eq.(33) can be used to compute ZPE provided that Λ/κ be considered as an eigenvalue of $\hat{\Lambda}_\Sigma$. In particular, Eq.(33) represents the Sturm-Liouville problem associated with the cosmological constant. To solve Eq.(33) is a quite impossible task. Therefore, we are oriented to use a variational approach with trial wave functionals. The related boundary conditions are dictated by the choice of the trial wave functionals which, in our case are of the Gaussian type. Different types of wave functionals correspond to different boundary conditions. The choice of a Gaussian wave functional is justified by the fact that ZPE should be described by a good candidate of the "*vacuum state*". To fix ideas, we choose the line element (6) as background metric with $g_1(E/E_P) = g_2(E/E_P) = 1$, namely MDRs do not distort the metric. Then we consider a perturbation of the metric tensor of the form $g_{ij} = \bar{g}_{ij} + h_{ij}$, where $\bar{g}_{ij}$ is the background metric and h_{ij} is a quantum fluctuation around the background. Thus Eq.(33) can be expanded in terms of h_{ij}. Since the kinetic part of $\hat{\Lambda}_\Sigma$ is quadratic in the momenta, we only need to expand the three-scalar curvature $\int d^3x\sqrt{g}^3R$ up to the quadratic order. To proceed with the computation, we need an orthogonal decomposition on the tangent space of 3-metric deformations[22]:

$$h_{ij} = \frac{1}{3}(\sigma + 2\nabla\cdot\xi)\,g_{ij} + (L\xi)_{ij} + h_{ij}^\perp. \tag{36}$$

The operator L maps ξ_i into symmetric tracefree tensors

$$(L\xi)_{ij} = \nabla_i\xi_j + \nabla_j\xi_i - \frac{2}{3}g_{ij}(\nabla\cdot\xi)\,, \tag{37}$$

$h_{ij}^\perp$ is the traceless-transverse component of the perturbation (TT), namely $g^{ij}h_{ij}^\perp = 0$, $\nabla^i h_{ij}^\perp = 0$ and h is the trace of h_{ij}. It is immediate to recognize that the trace element $\sigma = h - 2(\nabla\cdot\xi)$ is gauge invariant. The same decomposition can be done also on the momentum π^{ij} and induces the following transformation on the functional measure $\mathcal{D}h_{ij} \to \mathcal{D}h_{ij}^\perp\mathcal{D}\xi_i\mathcal{D}\sigma J_1$, where J_1 is the Jacobian related to the gauge vector variable ξ_i. The only physical information is encoded

$$\frac{1}{V}\frac{\left\langle \Psi^\perp \left| \int_\Sigma d^3x\left[\hat{\Lambda}_\Sigma^\perp\right]^{(2)} \right| \Psi^\perp \right\rangle}{\langle \Psi^\perp | \Psi^\perp \rangle} = -\frac{\Lambda^\perp}{\kappa}. \tag{38}$$

After having functionally integrated, we find

$$\hat{\Lambda}_\Sigma^\perp = \frac{1}{4V}\int_\Sigma d^3x\sqrt{\bar{g}}G^{ijkl}\left[(2\kappa)\,K^{-1\perp}(x,x)_{ijkl} + \frac{1}{(2\kappa)}\left(\tilde{\triangle}_L\right)^a_j K^\perp(x,x)_{iakl}\right], \tag{39}$$

---

[1] See also Ref.[21] for an application of the method to a $f(R)$ theory.

where

$$\left(\tilde{\triangle}_L h^{\perp}\right)_{ij} = \left(\triangle_L h^{\perp}\right)_{ij} - 4R_i^k h_{kj}^{\perp} + {}^3R h_{ij}^{\perp} \tag{40}$$

is the modified Lichnerowicz operator and $\triangle_L$is the Lichnerowicz operator defined by

$$(\triangle_L h)_{ij} = \triangle h_{ij} - 2R_{ikjl}h^{kl} + R_{ik}h_j^k + R_{jk}h_i^k \qquad \triangle = -\nabla^a \nabla_a. \tag{41}$$

G^{ijkl} represents the inverse DeWitt metric and all indices run from one to three. Note that the term $-4R_i^k h_{kj}^{\perp} + {}^3R h_{ij}^{\perp}$ disappears in four dimensions. The propagator $K^{\perp}(x,x)_{iakl}$ can be represented as

$$K^{\perp}(\overrightarrow{x},\overrightarrow{y})_{iakl} = \sum_{\tau} \frac{h_{ia}^{(\tau)\perp}(\overrightarrow{x})\, h_{kl}^{(\tau)\perp}(\overrightarrow{y})}{2\lambda(\tau)}, \tag{42}$$

where $h_{ia}^{(\tau)\perp}(\overrightarrow{x})$ are the eigenfunctions of $\tilde{\triangle}_L$. τ denotes a complete set of indices and $\lambda(\tau)$ are a set of variational parameters to be determined by the minimization of Eq.(39). The expectation value of $\hat{\Lambda}_{\Sigma}^{\perp}$ is easily obtained by inserting the form of the propagator into Eq.(39) and minimizing with respect to the variational function $\lambda(\tau)$. Thus the total one loop energy density for TT tensors becomes

$$\frac{\Lambda}{8\pi G} = -\frac{1}{2}\sum_{\tau}\left[\sqrt{\omega_1^2(\tau)} + \sqrt{\omega_2^2(\tau)}\right]. \tag{43}$$

The above expression makes sense only for $\omega_i^2(\tau) > 0$, where ω_i are the eigenvalues of $\tilde{\triangle}_L$. For a background of the form of Eq.(6), if we define the reduced fields $f_i(x) = F_i(x)/r$, we find that the Lichnerowicz operator $\left(\tilde{\triangle}_L h^{\perp}\right)_{ij}$ can be reduced to

$$\left[-\frac{d^2}{dx^2} + \frac{l(l+1)}{r^2} + m_i^2(r)\right] f_i(x) = \omega_{i,l}^2 f_i(x) \qquad i = 1,2 \quad , \tag{44}$$

where we have used the Regge-Wheeler representation[23] and $m_1^2(r)$ and $m_2^2(r)$ are two r-dependent effective masses

$$\begin{cases} m_1^2(r) = \frac{6}{r^2}\left(1 - \frac{b(r)}{r}\right) + \frac{3}{2r^2}b'(r) - \frac{3}{2r^3}b(r) \\ \\ m_2^2(r) = \frac{6}{r^2}\left(1 - \frac{b(r)}{r}\right) + \frac{1}{2r^2}b'(r) + \frac{3}{2r^3}b(r) \end{cases} \qquad (r \equiv r(x)). \tag{45}$$

In Eq.(44) , we have defined

$$dx = \pm\frac{dr}{\sqrt{1 - \frac{b(r)}{r}}}. \tag{46}$$

Like in the entropy calculation of section (II), we use the W.K.B. method and we define two r-dependent radial wave numbers

$$k_i^2(r,l,\omega_{i,nl}) = \omega_{i,nl}^2 - \frac{l(l+1)}{r^2} - m_i^2(r) \qquad i = 1,2 \quad . \tag{47}$$

For every degree of freedom of the graviton we apply Eq.(14) and we find that Eq.(43) becomes

$$\frac{\Lambda}{8\pi G} = -\frac{1}{\pi}\sum_{i=1}^{2}\int_0^{+\infty} \omega_i \frac{d\tilde{g}(\omega_i)}{d\omega_i} d\omega_i. \tag{48}$$

This is the one loop graviton contribution to the induced cosmological constant. The explicit evaluation of Eq.(48) gives

$$\frac{\Lambda}{8\pi G} = \rho_1 + \rho_2 = -\frac{1}{4\pi^2}\sum_{i=1}^{2}\int_{\sqrt{m_i^2(r)}}^{+\infty} \omega_i^2 \sqrt{\omega_i^2 - m_i^2(r)}\, d\omega_i, \tag{49}$$

where we have included an additional 4π coming from the angular integration. ρ_1 and ρ_2 are divergent and traditionally the use of the zeta function regularization keeps the divergences under control. To this purpose we reconsider Eq.(32) in presence of Gravity's Rainbow and we find[2]

$$\frac{g_2^3\left(E/E_P\right)}{\tilde{V}}\frac{\left\langle \Psi\left|\int_\Sigma d^3x\tilde{\Lambda}_\Sigma\right|\Psi\right\rangle}{\left\langle\Psi|\Psi\right\rangle}=-\frac{\Lambda}{\kappa}, \tag{50}$$

where

$$\tilde{\Lambda}_\Sigma=(2\kappa)\frac{g_1^2\left(E/E_P\right)}{g_2^3\left(E/E_P\right)}\tilde{G}_{ijkl}\tilde{\pi}^{ij}\tilde{\pi}^{kl}-\frac{\sqrt{\tilde{g}}\tilde{R}}{(2\kappa)\,g_2\left(E/E_P\right)} \tag{51}$$

The symbol "$\sim$" indicates the quantity computed in absence of rainbow's functions $g_1\left(E/E_P\right)$ and $g_2\left(E/E_P\right)$. Of course, Eqs.(50) and (51) reduce to the ordinary Eqs.(32, 33) and (35) when $E/E_{Pl}\to 0$. By repeating the procedure leading to Eq.(39), we find that the total one loop energy density becomes

$$\frac{\Lambda}{8\pi G}=-\frac{1}{3\pi^2}\sum_{i=1}^{2}\int_{E^*}^{+\infty}E_i g_1\left(E/E_P\right)g_2\left(E/E_P\right)\frac{d}{dE_i}\sqrt{\left(\frac{E_i^2}{g_2^2\left(E/E_P\right)}-m_i^2\left(r\right)\right)^3}dE_i, \tag{52}$$

where E^* is the value which annihilates the argument of the root. To further proceed, we choose a form of $g_1\left(E/E_P\right)$ and $g_2\left(E/E_P\right)$ suggested by a Noncommutative geometry analysis[10]. If we fix

$$g_1\left(E/E_P\right)=\left(1+\beta\frac{E}{E_P}\right)\exp(-\alpha\frac{E^2}{E_P^2})\qquad\text{and}\qquad g_2\left(E/E_P\right)=1, \tag{53}$$

with $\alpha>0$ and $\beta\in\mathbb{R}$, Eq.(52) can be easily integrated. However, it is more useful to give the asymptotic expansion for large and small x, where $x=\sqrt{m_{1,2}^2\left(r\right)/E_P^2}$. The asymptotic expansion for large x is

$$\frac{\Lambda}{8\pi G}\simeq-\frac{\left(2\beta\alpha^{3/2}+\sqrt{\pi}\alpha^2\right)x}{4\alpha^{7/2}}-\frac{8\beta\alpha^{5/2}+3\sqrt{\pi}\alpha^3}{16\alpha^{11/2}x}+\frac{3}{128}\frac{16\beta\alpha^{7/2}+5\sqrt{\pi}\alpha^4}{\alpha^{15/2}x^3}+O\left(x^{-4}\right), \tag{54}$$

while for small x, one gets

$$\frac{\Lambda}{8\pi G}\simeq-\frac{4\alpha^{5/2}+3\sqrt{\pi}\beta\alpha^2}{4\alpha^{9/2}}+O\left(x^3\right). \tag{55}$$

If we set

$$\beta=-\frac{\sqrt{\alpha\pi}}{2}, \tag{56}$$

then the linear divergent term of the asymptotic expansion (54) disappears. This means that $\Lambda/8\pi G$ vanishes for large x. On the other hand for small x we get

$$\frac{\Lambda}{8\pi G}\simeq\frac{3\pi-8}{8\alpha^2}+O\left(x^3\right), \tag{57}$$

where we have used the result of expansion (55). It is possible to show that with choice (56), the induced cosmological constant is always positive. Note that when $\beta=0$, the pure "*Gaussian*" choice can not give a positive induced cosmological constant[17].

---

[1] X. Li, *Phys. Lett.* **B 540**, 9 (2002), gr-qc/0204029.
[2] Z. Ren, W. Yue-Qin and Z. Li-Chun, Class. Quant. Grav. **20** (2003), 4885.

---

[2] Details of the calculation in presence of Gravity's Rainbow can be found in Ref.[17]

[3] G. Amelino-Camelia, *Class.Quant.Grav.* **23**, 2585 (2006), gr-qc/0506110. G. Amelino-Camelia, *Gen.Rel.Grav.* **33**, 2101 (2001), gr-qc/0106080.

[4] A.Kempf, G. Mangano and R.B. Mann, *Phys. Rev.* **D 52**, 1108 (1995), hep-th/9412167; L. J. Garay, *Int. J. Mod. Phys.* **A 10**, 145 (1995), gr-qc/9403008; D. V. Ahluwalia, *Phys. Lett.* **A 275**, 31 (2000), gr-qc/0002005.

[5] L. Susskind and J. Uglum, *Phys. Rev.* **D 50**, 2700 (1994). J.L.F. Barbon and R. Emparan, *Phys. Rev.* **D 52**, 4527 (1995), hep-th/9502155. E. Winstanley, *Phys. Rev.* **D 63**, 084013 (2001), hep-th/0011176.

[6] J.- G. Demers, R. Lafrance and R.C. Myers, *Phys. Rev.* **D 52**, 2245 (1995), gr-qc/9503003. D. V. Fursaev and S. N. Solodukhin, *Phys. Lett.* **B 365**, 51 (1996), hep-th/9412020. S. P. Kim, S. K. Kim, K.-S. Soh and Jae Hyung Yee, *Int. J. Mod. Phys.* **A 12**, 5223 (1997), gr-qc/9607019.

[7] F. Zhao and F. He, *Chin. Phys. Lett.* **27** (2010) 020401.

[8] E. Di Grezia, G. Esposito and G. Miele, *J. Phys.* **A 41**, 164063 (2008) [arXiv:0707.3318 [gr-qc]]; R. Banerjee, B. R. Majhi and S. Samanta, *Phys. Rev.* **D 77**, 124035 (2008) [arXiv:0801.3583 [hep-th]]. R. Casadio and P. Nicolini, *JHEP* **0811**, 072 (2008) [arXiv:0809.2471 [hep-th]]; Y. X. Chen and K. N. Shao, *Phys. Lett.* **B 678**, 131 (2009) [arXiv:0905.0948 [hep-th]].

[9] W. H. Huang and K. W. Huang, *Phys. Lett.* **B 670**, 416 (2009) [arXiv:0808.0324 [hep-th]];

[10] R. Garattini and P. Nicolini, *Phys. Rev.* **D 83**, 064021 (2011); arXiv:1006.5418 [gr-qc].

[11] H. Bai and M. L. Yan, JHEP **7** 58 (2003), gr-qc/0303006.

[12] G. Amelino-Camelia, *Int.J.Mod.Phys.* **D 11**, 35 (2002); gr-qc/0012051. G. Amelino-Camelia, *Phys.Lett.* **B 510**, 255 (2001); hep-th/0012238.

[13] J. L. Cortes and J. Gamboa, *Phys. Rev.* **D 71**, 065015 (2005); hep-th/0405285.

[14] G. Amelino-Camelia, M. Arzano, Y. Ling and G. Mandanici, *Class. Quant. Grav.* **23**, 2585 (2006); gr-qc/0506110.

[15] J. Magueijo and L. Smolin, *Class. Quant. Grav.* **21**, 1725 (2004) [arXiv:gr-qc/0305055].

[16] R. Garattini, *Phys. Lett.* **B 685**, 329 (2010) [arXiv:0902.3927 [gr-qc]].

[17] R. Garattini and G. Mandanici, *Phys. Rev.* **D 83**, 084021 (2011), arXiv:1102.3803 [gr-qc].

[18] G. 't Hooft, *Nucl. Phys.* **B 256**, 727 (1985).

[19] B. S. DeWitt, *Phys. Rev.* **160**, 1113 (1967).

[20] R. Garattini, *TSPU Vestnik* **44 N7**, 72 (2004); gr-qc/0409016.

[21] S. Capozziello and R. Garattini, *Class.Quant.Grav.* **24**, 1627 (2007); gr-qc/0702075.

[22] M. Berger and D. Ebin, *J. Diff. Geom.* **3**, 379 (1969). J. W. York Jr., *J. Math. Phys.*, **14**, 4 (1973); *Ann. Inst. Henri Poincaré* **A 21**, 319 (1974). P. O. Mazur and E. Mottola, *Nucl. Phys.* **B 341**, 187 (1990).D.V. Vassilevich, *Int. J. Mod. Phys.* **A 8**, 1637 (1993).

[23] T. Regge and J. A. Wheeler, *Phys. Rev.* **108**, 1063 (1957).